

Tensor product multiplicities and convex polytopes in partition space

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*Dedicated to I.M. Gelfand
on his 75th birthday*

Abstract. *We study multiplicities in the decomposition of tensor product of two irreducible finite dimensional modules over a semisimple complex Lie algebra. A conjectural expression for such multiplicity is given as the number of integral points of a certain convex polytope. We discuss some special cases, corollaries and confirmations of the conjecture.*

0. INTRODUCTION

In this paper we shall deal with irreducible finite-dimensional modules over a semisimple complex Lie algebra \mathfrak{g} . The fundamental role in theory of these modules and their numerous physical applications is played by two kinds of multiplicities. The first is the weight multiplicity $K_{\lambda\beta}$ of weight β in the irreducible \mathfrak{g} -module V_λ with highest weight λ . The second one is the tensor product multiplicity $c_{\lambda\nu}^\mu$ of V_μ in $V_\lambda \otimes V_\nu$. In fact, the weight multiplicity can be obtained as limit case of the tensor product multiplicity, so we shall mostly study multiplicities $c_{\lambda\nu}^\mu$.

Our main «result» is the conjectural expression for $c_{\lambda\nu}^\mu$ as the number of integral points in certain convex polytope. For $\mathfrak{g} = \mathfrak{sl}_n$ (or \mathfrak{gl}_n) such representation was given by I.M. Gelfand and one of the authors in [1], [2], [3]. Convex polytopes constructed there lie in the space of Gelfand-Tsetlin patterns. Here we present new approach replacing Gelfand-Tsetlin patterns by partitions into sum of positive roots. This language

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makes sense for any semisimple Lie algebra g .

Our approach comes back to the idea of F.A. Berezin and I.M. Gelfand [4], who emphasized the fundamental role of the hierarchy

$$P(\gamma) \rightarrow K_{\lambda\beta} \rightarrow c_{\lambda\nu}^{\mu},$$

where $P(\gamma)$ is the Kostant partition function [5]. The functions $P(\gamma)$, $K_{\lambda\beta}$ and $c_{\lambda\nu}^{\mu}$ depend respectively on 1, 2 and 3 weights, and we have the following expressions due to B. Kostant:

$$(0.1) \quad K_{\lambda,\beta} = \sum_{w \in W} \det w \cdot P(w(\lambda + \rho) - \beta - \rho),$$

$$(0.2) \quad c_{\lambda\nu}^{\mu} = \sum_{w \in W} \det w \cdot K_{\lambda, w(\mu + \rho) - \nu - \rho},$$

where W is the Weyl group of g , and ρ is the half-sum of positive roots. Let R be the root system of g , and R^+ the set of positive roots. By definition, $P(\gamma) = \text{card} \{ (m_{\alpha})_{\alpha \in R^+} : m_{\alpha} \in \mathbf{Z}_+, \sum_{\alpha} m_{\alpha} \cdot \alpha = \gamma \}$, i.e., $P(\gamma)$ is the number of partitions of γ into the sum of positive roots. This number is naturally interpreted as the number of integral points of the convex polytope $\Delta(\gamma)$ in the «partition» space \mathbf{R}^{R^+} with coordinates $(x_{\alpha})_{\alpha \in R^+}$. Namely, $\Delta(\gamma)$ is the intersection of the affine plane $\{ \sum_{\alpha} x_{\alpha} \cdot \alpha = \gamma \}$ of codimension $\text{rk } g$ with the «positive octant» $\{ x_{\alpha} \geq 0 \text{ for all } \alpha \}$. Integral points of $\Delta(\gamma)$ will be called g -partitions of weight γ .

It is well-known that

$$(0.3) \quad 0 \leq c_{\lambda\nu}^{\mu} \leq K_{\lambda, \mu - \nu} \leq P(\lambda + \nu - \mu)$$

(see §1 below). This suggests to look for expressions for multiplicities $K_{\lambda\beta}$ and $c_{\lambda\nu}^{\mu}$ also in terms of g -partitions. Our conjecture is that there are convex polytopes $\Delta_{\lambda\nu}^{\mu} \subset \Delta_{\lambda, \mu - \nu} \subset \Delta(\lambda + \nu - \mu)$ such that $c_{\lambda\nu}^{\mu}$ is that number of integral points of $\Delta_{\lambda\nu}^{\mu}$ (and similarly for $K_{\lambda, \mu - \nu}$).

Note that $P(\lambda + \nu - \mu)$ is the «leading term» corresponding to $w = e$ in alternating sum (0.1) for $K_{\lambda, \mu - \nu}$; similarly, $K_{\lambda, \mu - \nu}$ is the leading term in (0.2) for $c_{\lambda\nu}^{\mu}$. It follows that once the polytopes $\Delta_{\lambda\nu}^{\mu}$ and $\Delta_{\lambda, \mu - \nu}$ are found, one can in principle prove the conjecture using formulas (0.1) and (0.2) by cancelling alternating terms. This method was used in [6] for $g = sl_n$. In this case the expression for $c_{\lambda\nu}^{\mu}$ in terms of g -partitions (see §2 below) is equivalent to the one given in [1-3] and is also equivalent to the classical Littlewood-Richardson rule (see [7], [8]).

Now we give a more precise version of our conjecture. Let $\Pi \subset R^+$ be the set of simple roots of g , and ω_{α} ($\alpha \in \Pi$) be the fundamental weight corresponding to α .

CONJECTURE 0.1. *There are linear forms $\ell_{\alpha}^{(s)}$, $n_{\alpha}^{(t)}$ on \mathbf{R}^{R^+} (where $\alpha \in \Pi$ and for each α s and t run over certain finite sets) such that for every highest weights*

$\mu, \lambda = \sum_{\alpha \in \Pi} \ell_{\alpha} \omega_{\alpha}$, and $\nu = \sum_{\alpha \in \Pi} n_{\alpha} \omega_{\alpha}$ the multiplicity $c_{\lambda\nu}^{\mu}$ is equal to the number of g -partitions m of the weight $\lambda + \nu - \mu$ satisfying the inequalities $\max_s \ell_{\alpha}^{(s)}(m) \leq \ell_{\alpha}$, $\max_t n_{\alpha}^{(t)}(m) \leq n_{\alpha}$ for all $\alpha \in \Pi$.

Explicit (conjectural) expression of forms $\ell_{\alpha}^{(s)}$, $n_{\alpha}^{(t)}$ for all classical Lie algebras will be given in §2 (the case G_2 will be considered in another publication).

Conjecture 0.1 implies the following two expressions for weight multiplicities.

COROLLARY. *If $\ell_{\alpha}^{(s)}$, $n_{\alpha}^{(t)}$ are linear forms from Conjecture 0.1 then for any highest weight $\lambda = \sum_{\alpha \in \Pi} \ell_{\alpha} \omega_{\alpha}$ and any weight β the weight multiplicity $K_{\lambda\beta}$ is equal to the number of g -partitions m of weight $\lambda - \beta$ such that $\max_s \ell_{\alpha}^{(s)}(m) \leq \ell_{\alpha}$ for all α , and is also equal to the number of g -partitions m of weight $\lambda - \beta$ such that $\max_t n_{\alpha}^{(t)}(m) \leq \ell_{\alpha}$ for all α .* ■

Conjecture 0.1 is closely connected with the problem of constructing «good» bases in the modules V_{λ} (see [1-3]). To explain this relationship we need some definitions. Let $V_{\lambda}(\beta)$ be the weight subspace of V_{λ} of weight β . For any highest weight $\nu = \sum_{\alpha \in \Pi} n_{\alpha} \omega_{\alpha}$ let $V_{\lambda}(\beta, \nu) = \{v \in V_{\lambda}(\beta) : e_{\alpha}^{n_{\alpha}+1} v = 0 \text{ for } \alpha \in \Pi\}$, where e_{α} is the root vector of weight α . It is well-known (see e.g. [9]) that

$$(0.4) \quad c_{\lambda\nu}^{\mu} = \dim V_{\lambda}(\mu - \nu, \nu).$$

Following [1-3] we say that a basis B in V_{λ} is good if each subspace $V_{\lambda}(\beta, \nu)$ is spanned by a part of B .

CONJECTURE 0.2. *For any g and λ there exists a good basis in V_{λ} .* ■

This conjecture was independently given by I.M. Gelfand and A.V. Zelevinsky (see [10]) and K. Baclawski [11]. In fact in [11] the conjecture was stated as theorem but, unfortunately, the proof is incorrect. In [2] the conjecture is proven for $g = sl_n$ using the deep results of [12]. In [13] it is proven for $g = sp_4$.

It is easy to see that conjectures 0.1 and 0.2 together imply the following

CONJECTURE 0.3. *Let $\ell_{\alpha}^{(s)}$ and $n_{\alpha}^{(t)}$ be linear forms from Conjecture 0.1. Then for each highest weight λ there is a good basis $B = \{v_m\}$ in V_{λ} indexed by g -partitions m as follows:*

(1) *For any weight β the subset $B \cap V_{\lambda}(\beta)$ is indexed by g -partitions m of weight $\lambda - \beta$ satisfying inequalities $\max_s \ell_{\alpha}^{(s)}(m) \leq \ell_{\alpha}$ for all α .*

(2) *For any $\alpha \in \Pi$, $v_m \in B$ $\min\{n : e_{\alpha}^{n+1} v_m = 0\} = \max_t n_{\alpha}^{(t)}(m)$.* ■

It was explained in [2] and [13] that the notion of good basis is closely connected with the problem of canonical definition of Clebsch-Gordan coefficients (or equivalently of canonical definition of tensor operators, see e.g. [14] and references there). We hope that the parametrization of basis vectors by \mathfrak{g} -partitions as in Conjecture 0.3 will be helpful for physical applications.

There are two other general methods of computation of multiplicities. The first one comes back to D.E. Littlewood and is developed by B.G. Wybourne, R.C. King and others (see [15-18]). This is so called « S -function method» based on the theory of symmetric polynomials. This method enables one to express multiplicities $K_{\lambda\beta}$ and $c_{\lambda\nu}^{\mu}$ in terms of Young tableaux. Unfortunately, the answer in general is not completely combinatorial i.e., includes some negative terms (produced by so called modification rules). Thus, it is difficult to use this method for parametrization of basis vectors.

Another approach to multiplicities and special bases is so called «Standard monomial theory» (see [19] and references there). This method gives an unified combinatorial expression for weight multiplicities and also the special basis in modules V_{λ} «of classical type» ([19]). Note that this basis is not «good» in the sense of Conjecture 0.2.

Recently P. Littelmann [20] obtained the unified combinatorial expression for $c_{\lambda\nu}^{\mu}$ for all modules of classical type. His result is based on the standard monomial theory.

It would be very interesting to compare these two approaches to multiplicities with our approach.

The paper is organized as follows. In §1 we collect some general results on multiplicities. In particular, we show here that multiplicities in reduction of \mathfrak{g} -modules to a Levi subalgebra \mathfrak{g}' as well as tensor product multiplicities for semisimple part of \mathfrak{g}' are special cases of tensor product multiplicities for \mathfrak{g} .

In §2 we give the explicit form of Conjecture 0.1 for all classical Lie algebras (Conjecture 2.2). There are given also some special cases of Conjecture 2.2 which can be verified independently.

In §3 Conjecture 2.2 is proven for classical Lie algebras of small rank. For $\mathfrak{g} = \mathfrak{sp}_4$ we deduce it from the results of [13], and for \mathfrak{o}_4 , \mathfrak{o}_5 and \mathfrak{o}_6 use the exceptional isomorphisms $D_2 = A_1 \times A_1$, $B_2 = C_2$, $D_3 = A_3$.

Finally, in §4 some reduction multiplicities including weight multiplicities are studied. For any classical Lie algebra \mathfrak{g} we introduce the notion of a \mathfrak{g} -pattern generalizing famous Gelfand-Tsetlin patterns, and establish the correspondence between \mathfrak{g} -patterns and \mathfrak{g} -partitions from Conjecture 2.2. The notion of a \mathfrak{sp}_{2r} -pattern is essentially due to D.P. Zhelobenko (see [9]); \mathfrak{o}_{2r+1} - and \mathfrak{o}_{2r} -patterns seem to be new although they are closely connected with generalized Young tableaux considered in [15].

1. GENERAL PROPERTIES OF MULTIPLICITIES

Let \mathfrak{g} be a complex semisimple Lie algebra, \mathfrak{h} a Cartan subalgebra of \mathfrak{g} , $R \subset \mathfrak{h}^*$ the

root system. We fix the set of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$. Let $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}(\alpha)$ be the root decomposition of \mathfrak{g} . Choose standard generators $e_{\pm\alpha_i} \in \mathfrak{g}(\pm\alpha_i)$, $h_{\alpha_i} = [e_{\alpha_i}, e_{-\alpha_i}] \in \mathfrak{h}$ ($i = 1, \dots, r$). Let $\omega_1, \dots, \omega_r \in \mathfrak{h}^*$ be the fundamental weights of \mathfrak{g} defined by $\omega_i(h_{\alpha_j}) = \delta_{ij}$.

Recall that $\lambda \in \mathfrak{h}^*$ is a highest weight for \mathfrak{g} (i.e. the highest weight of an irreducible finite-dimensional \mathfrak{g} -module V_λ) if and only if $\lambda = \sum_{1 \leq i \leq r} \ell_i \omega_i$ with nonnegative integers ℓ_i . Integral linear combinations of fundamental weights are called *integral weights*; it is well-known that each weight of a finite-dimensional \mathfrak{g} -module is integral.

Now consider a weight subspace $V_\lambda(\beta) \subset V_\lambda$ and its subspace $V_\lambda(\beta, \nu)$ defined in §0.

PROPOSITION 1.1. *a) We have $0 \leq K_{\lambda\beta} \leq P(\lambda - \beta)$ (see §0). In particular, $V_\lambda(\beta) = 0$ unless $\lambda - \beta$ is a nonnegative integral linear combination of simple roots.*

b) We have $V_\lambda(\beta, \nu) = c_{\lambda\nu}^{\beta+\nu}$. In particular, $0 \leq c_{\lambda\nu}^{\beta+\nu} \leq K_{\lambda\beta}$. ■

This proposition is well-known (see e.g. [9]). Having it in mind it is natural to introduce the following notation: $c_{\lambda\nu}(\gamma) = \dim V_\lambda(\lambda - \gamma, \nu) = c_{\lambda\nu}^{\lambda+\nu-\gamma}$. If $\lambda = \sum_i \ell_i \omega_i$, $\nu = \sum_i n_i \omega_i$, $\gamma = \sum_i g_i \alpha_i$ then we shall sometimes write $c_{\ell_1 \dots \ell_r; n_1 \dots n_r}^{(g)}(g_1, \dots, g_r)$ for $c_{\lambda\nu}(\gamma)$. We shall extend the definition of $V_\lambda(\beta, \nu)$ (and so that of $c_{\lambda\nu}(\gamma)$) to the case when some of the coefficients n_i in decomposition $\nu = \sum n_i \omega_i$ may be equal to ∞ ; this simply means that the corresponding condition $e_{\alpha_i}^{n_i+1} v = 0$ is omitted. Since all e_{α_i} act on V_λ as nilpotent operators the infinite value of n_i can be replaced by sufficiently large positive value. Since $V_\lambda \otimes V_\nu \cong V_\nu \otimes V_\lambda$ it follows that $c_{\lambda\nu}(\gamma) = c_{\nu\lambda}(\gamma)$.

For each $\Pi' \subset \Pi$ we define the reductive Lie subalgebra $\mathfrak{g}' = \mathfrak{g}(\Pi') \subset \mathfrak{g}$ to be

$$\mathfrak{g}' = \mathfrak{h} \oplus \bigoplus_{\alpha} \mathfrak{g}(\alpha),$$

where the sum is over all roots α of the form $\sum_{\alpha_i \in \Pi'} c_i \alpha_i$. The subalgebras $\mathfrak{g}(\Pi')$ will be called Levi subalgebras of \mathfrak{g} . In particular, $\mathfrak{g}(\emptyset) = \mathfrak{h}$ is a Levi subalgebra. We denote by V'_β the irreducible \mathfrak{g}' -module with highest weight β .

PROPOSITION 1.2. *The multiplicity of $V'_{\lambda-\gamma}$ in the restriction $V_\lambda|_{\mathfrak{g}'}$ is equal to $c_{\lambda\nu}(\gamma)$, where the coefficient n_i in the decomposition $\nu = \sum n_i \omega_i$ is ∞ for $\alpha_i \notin \Pi'$, and 0 for $\alpha_i \in \Pi'$. In particular, the weight multiplicity $K_{\lambda, \lambda-\gamma}$ is equal to $c_{\lambda\nu}(\gamma)$, where all coefficients n_i of ν are ∞ .*

This follows at once from definitions since the subspace of highest vectors of weight $\lambda - \gamma$ in $V_\lambda|_{\mathfrak{g}'}$ is just $V_\lambda(\lambda - \gamma, \nu) = \{v \in V_\lambda(\lambda - \gamma) : e_\alpha v = 0 \text{ for } \alpha \in \Pi'\}$. ■

With a subset $\Pi' \subset \Pi$ we also associate the semisimple Lie subalgebra $\mathfrak{g}_0 = \mathfrak{g}_0(\Pi')$ generated by $e_{\pm\alpha}, h_\alpha$ for $\alpha \in \Pi'$. The Cartan subalgebra \mathfrak{h}_0 of \mathfrak{g}_0 is spanned by h_α for $\alpha \in \Pi'$, so the natural projection $p : \mathfrak{h}^* \rightarrow \mathfrak{h}_0^*$ sends $\omega_i (\alpha_i \notin \Pi')$ to 0. Without loss of generality we can assume that $\Pi' = \{\alpha_1, \dots, \alpha_k\}$ for some $k \leq r$.

PROPOSITION 1.3. *Let λ, ν be highest weights for \mathfrak{g} , and γ the weight of the form $\sum_{1 \leq i \leq k} g_i \alpha_i$. Let $\lambda' = p(\lambda), \nu' = p(\nu), \gamma' = p(\gamma)$ be the corresponding \mathfrak{g}_0 -weights.*

Then $c_{\lambda\nu}(\gamma) = c_{\lambda'\nu'}(\gamma')$. In other words, $c_{\ell_1 \dots \ell_r; n_1 \dots n_r}^{(\mathfrak{g})}(g_1, \dots, g_k, 0, \dots, 0)$ is independent on $\ell_{k+1}, \dots, \ell_r, n_{k+1}, \dots, n_r$ and equals $c_{\ell_1 \dots \ell_k; n_1 \dots n_k}^{(\mathfrak{g}_0)}(g_1, \dots, g_k)$.

Proof. Let $V_{\lambda'}$ denote the irreducible \mathfrak{g}_0 -module with highest weight λ' , let $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$, where \mathfrak{n}_0^+ (resp. \mathfrak{n}_0^-) is generated by $e_{\alpha_1}, \dots, e_{\alpha_k}$ (resp. $e_{-\alpha_1}, \dots, e_{-\alpha_k}$), and $\mathfrak{n}' = \bigoplus_{\alpha} \mathfrak{g}(\alpha)$, the sum over positive roots α involving some of the roots $\alpha_{k+1}, \dots, \alpha_r$. Our proposition follows readily from the next statement:

The \mathfrak{g}_0 -module $(V_{\lambda'})^{\mathfrak{n}'}$ of \mathfrak{n}' -invariants in $V_{\lambda'}$ is isomorphic to $V_{\lambda'}$ and this isomorphism identifies the weight subspace $V_{\lambda'}(\lambda' - \gamma')$ with $[V_{\lambda'}(\lambda - \gamma)]^{\mathfrak{n}'}$.

The isomorphism of $(V_{\lambda'})^{\mathfrak{n}'}$ with $V_{\lambda'}$ is clear from the fact that $(V_{\lambda'})^{\mathfrak{n}'}$ as a \mathfrak{g}_0 -module has the unique highest vector. The inclusion $[V_{\lambda'}(\lambda - \gamma)]^{\mathfrak{n}'} \subset V_{\lambda'}(\lambda' - \gamma')$ is evident, and converse inclusion follows from the equality $V_{\lambda'}(\lambda' - \gamma') = U^- V_{\lambda'}(\lambda')$, where U^- is the weight subspace $U(\mathfrak{n}_0^-)(-\gamma')$ of the universal enveloping algebra $U(\mathfrak{n}_0^-)$. ■

Proposition 1.3 shows that tensor product multiplicities for «standard» semisimple Lie subalgebras of \mathfrak{g} are special cases of these multiplicities for \mathfrak{g} .

Next proposition will be used as a test to our conjectures on multiplicities.

PROPOSITION 1.4. *Let \mathfrak{g} be simple, and θ be the maximal root of \mathfrak{g} ([21]). Then $c_{\theta\nu}^{\nu} = c_{\theta\nu}(\theta)$ is equal to the number of nonzero coefficients n_i in the expansion $\nu = \sum n_i \omega_i$.*

Proof. By definition, $V_\theta \cong \mathfrak{g}$ is the adjoint \mathfrak{g} -module, and $V_\theta(0) = \mathfrak{h}$. Therefore, $c_{\theta\nu}(\theta) = \dim V_\theta(0, \nu) = \dim \{h \in \mathfrak{h} : (\text{ad } e_{\alpha_i})^{n_i+1}(h) = 0, i = 1, \dots, r\}$. But $\text{ad } e_{\alpha_i}(h) = \alpha_i(h) \cdot e_{\alpha_i}$, and $(\text{ad } e_{\alpha_i})^2(h) = 0$, which implies our statement. ■

2. MAIN CONJECTURE

In this section we give a precise form of Conjecture 0.1 for all classical simple Lie algebras. We use the standard numeration of simple roots $\alpha_1, \dots, \alpha_r$ and corresponding fundamental weights $\omega_1, \dots, \omega_r$ (see [21]). Thus, the integral weights for each type B_r, C_r, D_r will be written as linear combinations of standard basis vectors $\varepsilon_1, \dots, \varepsilon_r$

in \mathbb{R}^r (with integer or half-integer coefficients); for A_{r-1} the lattice of integral weights is identified with $\mathbb{Z}^r / \mathbb{Z}(\varepsilon_1 + \dots + \varepsilon_r)$.

Let $m_{ij} = m_{\varepsilon_i - \varepsilon_j}$, $m_{ij}^+ = m_{\varepsilon_i + \varepsilon_j}$, $m_i = m_{\varepsilon_i}$, so a o_{2r+1} -partition of weight $\gamma \in \mathbb{Z}^r$ is a vector $m = (m_{ij}(1 \leq i < j \leq r), m_{ij}^+(1 \leq i < j \leq r), m_i(1 \leq i \leq r))$ such that all its components are nonnegative integers, and

$$\sum m_{ij}(\varepsilon_i - \varepsilon_j) + \sum m_{ij}^+(\varepsilon_i + \varepsilon_j) + \sum m_i \varepsilon_i = \gamma.$$

It will be convenient for our purposes to realize sl_r -, sp_{2r} - and o_{2r} -partitions in terms of o_{2r+1} -partitions. Namely, we identify an sl_r -partition with an o_{2r+1} -partition (m_{ij}^+, m_{ij}, m_i) such that $m_i = m_{ij}^+ = 0$ for all i, j . Similarly, an o_{2r} -partition is an o_{2r+1} -partition (m_{ij}, m_{ij}^+, m_i) such that $m_i = 0$ for all i . Finally, a sp_{2r} -partition is identified with a o_{2r+1} -partition (m_{ij}, m_{ij}^+, m_i) such that all m_i are even. It is clear that such identification preserves the weight of a partition.

Now we give an expression for

$$c_{\lambda\nu}^{\lambda+\nu-\gamma} = c_{\lambda\nu}(\gamma) = c_{\varepsilon_1 \dots \varepsilon_{r-1}; n_1 \dots n_{r-1}}^{(g)}(g_1, \dots, g_{r-1}) \quad (\text{see } \S 1)$$

for $g = sl_r$. For any sl_r -partition $m = (m_{ij})$ and any pair (i, j) put

$$(2.1) \quad \Delta_{ij} = \Delta_{ij}(m) = m_{ij} - m_{i+1, j+1}$$

(with the convention that $m_{ij} = 0$ unless $1 \leq i < j \leq r$, so for example $\Delta_{0, j} = -m_{1, j+1}$, $\Delta_{i, r} = m_{i, r}$). We define the linear forms $\ell_j^{(s)} = \ell_j^{(s)}(m)$, $n_i^{(t)} = n_i^{(t)}(m)$ ($1 \leq i, j \leq r-1$) by

$$(2.2) \quad \ell_j^{(s)} = - \sum_{0 \leq p \leq s} \Delta_{pj} \quad (0 \leq s < j); \quad n_i^{(t)} = \sum_{t \leq p \leq r} \Delta_{ip} \quad (i < t \leq r).$$

The following theorem is the first motivation of Conjecture 0.1.

THEOREM 2.1. *The multiplicity $c_{\varepsilon_1 \dots \varepsilon_{r-1}; n_1 \dots n_{r-1}}^{(sl_r)}(g_1, \dots, g_{r-1})$ is equal to the number of sl_r -partitions m of weight $\sum_{1 \leq i \leq r-1} g_i(\varepsilon_i - \varepsilon_{i+1})$ such that $\max_s \ell_j^{(s)}(m) \leq \ell_j$, $\max_t n_i^{(t)}(m) \leq n_i$ for all i, j .*

The proof will be given in §4.

REMARKS. 1. Let $\lambda, \mu, \nu \in \mathbb{Z}^r / \mathbb{Z}(\varepsilon_1 + \dots + \varepsilon_r)$ be three highest weights for sl_r . Choose representatives $\bar{\lambda}, \bar{\mu}, \bar{\nu} \in \mathbb{Z}_+^r$ of these weights so that $|\bar{\mu}| = |\bar{\lambda}| + |\bar{\nu}|$, where $|\mu_1 \varepsilon_1 + \dots + \mu_r \varepsilon_r| = \mu_1 + \dots + \mu_r$. Denote by $LR_{\bar{\lambda}\bar{\nu}}^{\bar{\mu}}$ the coefficient given by the

Littlewood-Richardson rule (see [8]), i.e. the tensor product multiplicity for \mathfrak{gl}_r . Then it is clear that $c_{\lambda\nu}^\mu = LR_{\lambda\nu}^\mu$.

2. Theorem 2.1 makes the statement of Proposition 1.3 for $\mathfrak{g} = \mathfrak{sl}_{r+1}, \mathfrak{so} = \mathfrak{sl}_r$ evident: since $g_r = m_{1,r+1} + m_{2,r+1} + \dots + m_{r,r+1}$, and all $m_{i,j}$ are nonnegative, it follows that $g_r = 0$ implies $m_{1,r+1} = \dots = m_{r,r+1} = 0$, and we are left with an \mathfrak{sl}_r -partition.

3. Consider the linear involution $\beta \rightarrow \hat{\beta}$ on integral \mathfrak{sl}_r -weights such that $\hat{\varepsilon}_i = -\varepsilon_{r+1-i}$, and extend it to \mathfrak{sl}_r -partitions by $\hat{m}_{ij} = m_{r+1-j, r+1-i}$. By definitions, $n_i^{(t)}(m) = \ell_{r-i}^{(r-t)}(\hat{m})$ for all i, t . Therefore, Theorem 2.1 implies $c_{\lambda\nu}(\gamma) = c_{\hat{\nu}\hat{\lambda}}(\hat{\gamma})$ (this equality follows at once from the tensor product interpretation of multiplicities and the fact that $V_{\hat{\lambda}} = (V_\lambda)^*$, the dual module).

Now consider other classical Lie algebras of types B_r, C_r and D_r . Let $I = I_r = \{1, 2, \dots, r, \bar{0}, \bar{1}, \dots, \bar{r}\}$ be linearly ordered by $\bar{0} < 1 < \bar{1} < 2 < \bar{2} < \dots < r < \bar{r}$. For each $s, t \in I$ define the linear form $\Delta_{s,t}$ on o_{2r+1} -partitions $m = (m_{ij}, m_{ij}^+, m_i)$ by

$$(2.3) \quad \begin{aligned} \Delta_{ij} &= m_{ij} - m_{ij}^+, & \Delta_{i\bar{j}} &= \Delta_{i+1, j+1}, \\ \Delta_{i\bar{j}} &= \Delta_{i\bar{j}} = \begin{cases} m_{i, j+i}^+ - m_{i+1, j+1} & \text{for } j < r \\ m_i - m_{i+1} & \text{for } j = r \end{cases} \end{aligned}$$

(with the convention that $m_{ij} = m_{ij}^+ = 0$ unless $1 \leq i < j \leq r$, and $m_i = 0$ unless $1 \leq i \leq r$).

For $i, j = 1, 2, \dots, r-1, s, t \in I$ and $\varepsilon = 0, 1$ define the linear forms $\ell_j^{(s)}, n_i^{(t, \varepsilon)}$ by formulas

$$(2.4) \quad \begin{cases} \ell_j^{(s)} = -\sum_{\bar{0} \leq p \leq s} \Delta_{pj}, \\ n_i^{(t, 0)} = \Delta_{i\bar{i}} + \sum_{i+1 \leq p \leq t} \Delta_{ip}, \\ n_i^{(t, 1)} = n_i^{(r, 0)} + \sum_{t \leq p \leq r} \Delta_{ip}, \end{cases}$$

where in each sum p runs over the indicated interval in I , and empty sums are understood as 0.

We shall also need the involution $m \rightarrow \hat{m}$ on the set of all o_{2r} -partitions defined by

$$(2.5) \quad \hat{m}_{ij} = \begin{cases} m_{ij} & \text{for } j < r \\ m_{ij}^+ & \text{for } j = r, \end{cases} \quad \hat{m}_{ij}^+ = \begin{cases} m_{ij}^+ & \text{for } j < r \\ m_{ij} & \text{for } j = r. \end{cases}$$

We define the forms $\ell_r^{(s)}$ ($s \in I$) and $n_r^{(r, 1)}$ separately in each of the cases $B_r,$

$C_r, D_r :$

$$\begin{aligned}
 D_r : \quad \ell_r^{(s)}(m) &= \ell_{r-1}^{(s)}(\hat{m}), & n_r^{(r,1)}(m) &= n_{r-1}^{(r,1)}(\hat{m}) = m_{r-1}^+, \\
 B_r : \quad \ell_r^{(s)} &= - \sum'_{\bar{0} \leq p \leq s} \Delta_{pr}, & n_r^{(r,1)}(m) &= m_r \\
 C_r : \quad \ell_r^{(s)} &= -\frac{1}{2} \sum'_{\bar{0} \leq p \leq s} \Delta_{pr}, & n_r^{(r,1)}(m) &= m_r/2.
 \end{aligned}
 \tag{2.6}$$

(here \sum' means that the summands $\Delta_{1,r}, \Delta_{2,r}, \dots$ are taken with coefficient 2).

MAIN CONJECTURE 2.2. *Let g be one of the classical simple Lie algebras of type $B_r, C_r,$ or D_r (i.e. $g = o_{2r+1}, sp_{2r}$ or o_{2r}). Then $c_{\ell_1 \dots \ell_r; n_1 \dots n_r}^{(g)}(g_1, \dots, g_r)$ is equal to the number of g -partitions m of weight $\sum g_i \alpha_i$ such that $\max_j \ell_j^{(s)}(m) \leq \ell_j, \max_{t, \varepsilon} n_i^{(t, \varepsilon)}(m) \leq n_i$ for all $i, j = 1, \dots, r$, where maximums are over all possible s, t, ε with the only exception that in case D_r the forms $n_i^{(t, \varepsilon)}$ ($1 \leq i \leq r-1$) equal to $n_i^{(r,0)}$ are not taken into account.*

REMARKS. 1. Actually, the linear forms contributing to Conjecture 2.2 are the following:

$$\begin{aligned}
 B_r, C_r : & \ell_j^{(s)} \quad (1 \leq j \leq r, \bar{0} \leq s < j), \\
 & n_i^{(t, \varepsilon)} \quad (1 \leq i \leq r-1, \bar{i} \leq t \leq r \text{ for } \varepsilon = 0, \text{ and } \bar{i} < t \leq r \text{ for } \varepsilon = 1), \quad n_r^{(r,1)}. \\
 D_r : & \ell_j^{(s)} \quad (1 \leq j \leq r-1, \bar{0} \leq s < j), \quad \ell_r^{(s)} \quad (\bar{0} \leq s < r-1), \\
 & n_i^{(t, \varepsilon)} \quad (1 \leq i \leq r-2, \bar{i} \leq t \leq r-1 \text{ for } \varepsilon = 0, \text{ and } \bar{i} < t \leq r \text{ for } \varepsilon = 1), \\
 & n_{r-1}^{(r,1)}, \quad n_r^{(r,1)}.
 \end{aligned}$$

2. Conjecture 2.2 is compatible with Proposition 1.3 in the same way as Theorem 2.1 (cf. Remark 2 to the Theorem 2.1). Namely, putting $g_r = 0$ we obtain the multiplicity for $s\ell_r$ predicted by Proposition 1.3. Similarly, putting $g_1 = 0$ we obtain the multiplicity for the subalgebra $g_0 (\{\alpha_2, \dots, \alpha_r\})$ i.e. the previous member of the same series as g .

3. It is easy to see that the expression for $c_{\lambda, \nu}(\gamma)$ given by Conjecture 2.2 in case C_r can be reformulated as follows:

$$c_{\ell_1 \dots \ell_r; n_1 \dots n_r}^{(sp_{2r})} (g_1, \dots, g_r)$$

is equal to the number of o_{2r+1} -partitions $m = (m_{ij}, m_{ij}^+, m_i)$ contributing to

$$c_{\ell_1 \dots \ell_{r-1}, 2\ell_r; n_1 \dots n_{r-1}, 2n_r}^{(o_{2r+1})} (g_1, \dots, g_{r-1}, 2g_r)$$

and such that all m_i are even.

4. In case D_r let $\lambda \rightarrow \hat{\lambda}$ be the linear involution on weights such that $\hat{\varepsilon}_i = \varepsilon_i$ for $i \leq r - 1$, $\hat{\varepsilon}_r = -\varepsilon_r$ (clearly, it is compatible with the involution $m \rightarrow \hat{m}$ on o_{2r} -partitions defined by (2.5)). Then Conjecture 2.2 easily implies $c_{\lambda\nu}^\mu = c_{\hat{\lambda}\hat{\nu}}^{\hat{\mu}}$ (which is of course evident since the involution $\lambda \rightarrow \hat{\lambda}$ is an automorphism of the root system D_r).

Applications of Theorem 2.1 and Conjecture 2.2 to Lie algebras of small rank will be given in §3. Now we give some confirmations of Conjecture 2.2. First we show that the Theorem 2.1 and Conjecture 2.2 imply Proposition 1.4. It suffices to prove

PROPOSITION 2.3. *Let g be one of classical simple Lie algebras of rank r , and $\theta = \sum_{1 \leq i \leq r} n_i \omega_i$ be its maximal root. Then there are exactly r g -partitions $m^{(1)}, \dots, m^{(r)}$ of weight θ satisfying $\max_i n_i^{(t)}(m^{(k)}) \leq n_i (1 \leq i \leq r)$, and they can be numerated in such a way that $\max_j \ell_j^{(s)}(m^{(k)}) = \delta_{jk}$ (where the $\ell_j^{(s)}$ and $n_i^{(t)}$ are linear forms from Theorem 2.1 and Conjecture 2.2).*

The proof is straightforward. We shall only give the list of $m^{(1)}, \dots, m^{(r)}$ (all parts not mentioned are understood to be 0).

$$A_r : \theta = \varepsilon_1 - \varepsilon_{r+1} = \omega_1 + \omega_r;$$

$$m^{(k)} = (m_{1,k+1} = m_{k+1,k+2} = m_{k+2,k+3} = \dots = m_{r,r+1} = 1).$$

$$B_r : \theta = \varepsilon_1 + \varepsilon_2 = \omega_2; \quad m^{(1)} = (m_{1,2} = m_{23} = m_{23}^+ = 1),$$

$$m^{(k)} = (m_{1,k+1} = m_{2,k+1}^+ = 1) (2 \leq k \leq r - 1),$$

$$m^{(r)} = (m_1 = m_2 = 1).$$

$$C_r : \theta = 2\varepsilon_1 = 2\omega_1; \quad m^{(k)} = (m_{1,k+1} = m_{1,k+1}^+ = 1) (1 \leq k \leq r - 1),$$

$$m^{(r)} = (m_1 = 2).$$

$$D_r (r \geq 4) : \theta = \varepsilon_1 + \varepsilon_2 = \omega_2; \quad m^{(1)}, \dots, m^{(r-1)} \text{ are the same as for } B_r,$$

$$m^{(r)} = (m_{1,r}^+ = m_{2,r} = 1).$$

PROPOSITION 2.4. *Conjecture 2.2 is true for the case when $\ell_2 = \ell_3 = \dots = \ell_r = 0$.*

The proof is based on the «involution method» of [6]. We postpone it till another publication.

Finally, consider the case $n_1 = n_2 = \dots = n_{r-1} = 0$.

COROLLARY 2.5 (OF CONJECTURE 2.2). *a) For $\ell_r \geq \min(n_r, 2(g_r - g_{r-1}))$ we have*

$$\begin{aligned} c_{\ell_1 \dots \ell_r; 00 \dots 0 n_r}^{(o_{2r+1})}(g_1, \dots, g_r) &= \\ &= \sum c_{\ell_1 \dots \ell_{r-1}; m_2 - m_1, \dots, m_r - m_{r-1}}^{(s\ell_r)}(g_1 - m_1, g_2 - m_1 - m_2, \dots, g_{r-1} - \\ &\quad - m_1 - \dots - m_{r-1}) \end{aligned}$$

the sum over all $(m_1, \dots, m_r) \in \mathbb{Z}^r$ such that $0 \leq m_1 \leq \dots \leq m_r \leq n_r$ and $m_1 + \dots + m_r = g_r$.

b) For $\ell_r \geq \min(n_r, 2g_r - g_{r-1})$ we have

$$\begin{aligned} c_{\ell_1 \dots \ell_r; 00 \dots 0 n_r}^{(sp_{2r})}(g_1, \dots, g_r) &= \\ &= \sum c_{\ell_1 \dots \ell_{r-1}; m_2 - m_1, \dots, m_r - m_{r-1}}^{(s\ell_r)}(g_1 - m_1, g_2 - m_1 - m_2, \dots, g_{r-1} - \\ &\quad - m_1 - \dots - m_{r-1}) \end{aligned}$$

the sum over all $(m_1, \dots, m_r) \in \mathbb{Z}^r$ such that $0 \leq m_1 \leq \dots \leq m_r \leq 2n_r$, $m_1 + \dots + m_r = 2g_r$, and m_1, \dots, m_r are even.

c) For $\ell_r \geq \min(n_r, g_r + g_{r-1} - g_{r-2})$ we have

$$\begin{aligned} c_{\ell_1 \dots \ell_r; 00 \dots 0 n_r}^{(o_{2r})}(g_1, \dots, g_r) &= \\ &= \sum c_{\ell_1 \dots \ell_{r-1}; m_2 - m_1, \dots, m_r - m_{r-1}}^{(s\ell_r)}(g_1 - m_1, \dots, g_{r-2} - m_1 - \dots - \\ &\quad - m_{r-2}, g_{r-1} - \frac{m_1 + \dots + m_{r-2}}{2}), \end{aligned}$$

the sum over all $(m_1, \dots, m_r) \in \mathbb{Z}^r$ such that $0 \leq m_1 \leq \dots \leq m_r \leq n_r$, $m_1 + \dots + m_r = 2g_r$ and $m_r = m_{r-1}, \dots, m_{r-2k} = m_{r-2k-1}$ (with the convention $m_0 = 0$).

We shall outline the proof of a). Consider an o_{2r+1} -partition $m = (m_{ij}, m_{ij}^+, m_i)$ contributing to $c_{\ell_1 \dots \ell_r; 00 \dots 0 n_r}^{(o_{2r+1})}(g_1, \dots, g_r)$ by Conjecture 2.2. It is easy to see that the inequalities $\max_{t, \epsilon} n_i^{(t, \epsilon)}(w) \leq n_i$ imply that $m_{ij}^+ = 0$ for all i, j , and $0 \leq m_1 \leq \dots \leq m_r \leq n_r$. Furthermore, the condition on weight of m implies $\sum m_i = g_r$. Now fix m_1, \dots, m_r satisfying these conditions and consider the remaining part (m_{ij}) of

m as an sl_r -partition m' . One can verify that the conditions of Conjecture 2.2 except those related to ℓ_r are equivalent to the fact that m' contributes to

$$c_{\ell_1, \dots, \ell_{r-1}; m_2 - m_1, \dots, m_r - m_{r-1}}^{(sl_r)}(g_1 - m_1, g_2 - m_1 - m_2, \dots, g_{r-1} - m_1 - \dots - m_{r-1})$$

by Theorem 2.1. It remains to observe that the inequality $\max_s \ell_r^{(s)}(m) \leq \ell_r$ follows from each one of the inequalities $\ell_r \geq n_r$ and $\ell_r \geq 2(g_r - g_{r-1})$. The proof of b) is entirely similar, and the proof of c) follows the same lines although is more complicated: in this case the parameters m_i are chosen so that $m_{r-2k} = m_{r-2k-1} = m_{r-2k-1, r}^+$.

Consider now the special case of Corollary 2.5 when $n_r = +\infty$. According to Proposition 1.2 the multiplicity in question is the reduction multiplicity from $g = o_{2r+1}, sp_{2r}$ or o_{2r} to the Levi subalgebra gl_r .

Let $\lambda = \lambda_1 \varepsilon_1 + \dots + \lambda_r \varepsilon_r$, $\beta = \beta_1 \varepsilon_1 + \dots + \beta_r \varepsilon_r$ be partitions in sense of [8], i.e. $\lambda_i, \beta_i \in \mathbb{Z}$, $\lambda_1 \geq \dots \geq \lambda_r \geq 0$, $\beta_1 \geq \dots \geq \beta_r \geq 0$. Let V_λ be the irreducible g -module with highest weight λ (note that for $g = o_{2r+1}$ and o_{2r} this excludes spinor modules), and V'_β the gl_r -module with highest weight β .

PROPOSITION 2.6. *For $g = o_{2r+1}$ the multiplicity of V'_β in $V_\lambda|_{gl_r}$ is equal to $\sum_\mu LR_{\beta\mu}^\lambda$ (see Remark 1 to the Theorem 2.1), where μ runs over all partitions $\mu_1 \geq \dots \geq \mu_r \geq 0$ such that $|\mu| = |\lambda| - |\beta|$. For $g = sp_{2r}$ this multiplicity is equal to $\sum_\nu LR_{\beta\nu}^\lambda$, the sum over all partitions ν with even parts such that $|\nu| = |\lambda| - |\beta|$. Finally, for $g = o_{2r}$ the multiplicity is equal to $\sum_{\nu'} LR_{\beta\nu'}^\lambda$, the sum over the same partitions ν as above, and ν' stands for the conjugate partition [8].*

This follows at once from Proposition 1.2, Corollary 2.5 and the well-known symmetry $LR_{\lambda\hat{\beta}}^\mu = LR_{\beta\mu}^\lambda$, where $\beta \rightarrow \hat{\beta}$ is the involution defined in Remark 3 to Theorem 2.1 (the coefficient $LR_{\lambda\hat{\beta}}^\mu$ should be understood as the multiplicity of V'_μ in $V'_\lambda \otimes V'_\beta$). On the other hand, Proposition can be deduced from the results of [17], which gives an independent affirmation of Conjecture 2.2.

Using Corollary 2.5 for $\ell_1 = \ell_2 = \dots = \ell_{r-1} = 0$, $\ell_r = +\infty$ and Proposition 1.2 we obtain

COROLLARY 2.7 (OF CONJECTURE 2.2). *Let g be one of the algebras o_{2r+1}, sp_{2r} , or o_{2r} . For any $n \geq 0$ the restriction of the g -module $V_{n\omega_r}$ to the Levi subalgebra gl_r is isomorphic to $\bigoplus_\beta V'_\beta$, where $\beta = \beta_1 \varepsilon_1 + \dots + \beta_r \varepsilon_r$ runs over all weights satisfying:*

- a) For $g = o_{2r+1}$ $n/2 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_r \geq -n/2$, $n/2 - \beta_i \in \mathbb{Z}$.
- b) For $g = sp_{2r}$ $n \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_r \geq -n$, $n - \beta_i \in 2\mathbb{Z}$.
- c) For $g = o_{2r}$ $n/2 \geq \beta_1 \geq \beta_2 \geq \dots \geq \beta_r \geq -n/2$, $n/2 - \beta_i \in \mathbb{Z}$, $\beta_r = \beta_{r-1}, \dots, \beta_{r-2k} = \beta_{r-2k-1}$ (with the convention $\beta_0 = n/2$).

REMARK. Part a) is equivalent to [8, ch. I, §5, Ex. 16, formula (2)].

3. CLASSICAL LIE ALGEBRAS OF SMALL RANK

The next theorem summarizes the statements of Theorem 1.2 and Conjecture 2.2 in some small rank cases.

THEOREM 3.1. *In each of the following cases the multiplicity $c_{\ell_1 \dots \ell_r; m_1 \dots m_r}^{(g)}$ (g_1, \dots, g_r) is equal to the number of g -partitions m satisfying the conditions:*

a) $A_1, g = sl_2 :$

$$(3.1) \left\{ \begin{array}{ll} \ell_1 \geq m_{12} & n_1 \geq m_{12} \\ & g_1 = m_{12} \end{array} \right.$$

b) $A_2, g = sl_3 :$

$$(3.2) \left\{ \begin{array}{ll} \ell_1 \geq m_{12} & n_1 \geq m_{13} \\ \ell_2 \geq m_{13} & n_1 \geq m_{13} + m_{12} - m_{23} \\ \ell_2 \geq m_{13} + m_{23} - m_{12} & n_2 \geq m_{23} \\ & g_1 = m_{12} + m_{13} \\ & g_2 = m_{23} + m_{13} \end{array} \right.$$

c) $A_3, g = sl_4 :$

$$(3.3) \left\{ \begin{array}{ll} \ell_1 \geq m_{12} & n_1 \geq m_{14} \\ \ell_2 \geq m_{13} & n_1 \geq m_{14} + m_{13} - m_{24} \\ \ell_2 \geq m_{13} + m_{23} - m_{12} & n_1 \geq m_{14} + m_{13} - m_{24} + m_{12} - m_{23} \\ \ell_3 \geq m_{14} & n_2 \geq m_{24} \\ \ell_3 \geq m_{14} + m_{24} - m_{13} & n_2 \geq m_{24} + m_{23} - m_{34} \\ \ell_3 \geq m_{14} + m_{24} - m_{13} + m_{34} - m_{23} & n_3 \geq m_{34} \\ & g_1 = m_{12} + m_{13} + m_{14} \\ & g_2 = m_{23} + m_{13} + m_{14} + m_{24} \\ & g_3 = m_{34} + m_{24} + m_{14} \end{array} \right.$$

d) $B_2, g = o_5 :$

$$(3.4) \left\{ \begin{array}{ll} \ell_1 \geq m_{12} & n_1 \geq m_{12}^+ \\ \ell_2 \geq m_1 & n_2 \geq m_{12}^+ + m_1 - m_2 \\ \ell_2 \geq m_1 + 2(m_{12}^+ - m_{12}) & n_1 \geq m_{12} + m_1 - m_2 \\ \ell_2 \geq m_2 + 2(m_{12}^+ - m_{12}) & n_2 \geq m_2 \\ & g_1 = m_{12} + m_{12}^+ + m_1 \\ & g_2 = 2m_{12}^+ + m_1 + m_2 \end{array} \right.$$

e) $C_2, g = sp_4 : (m_1, m_2 \text{ are even})$

$$(3.5) \left\{ \begin{array}{l} \underline{\ell_1 \geq m_{12}} \\ \ell_2 \geq m_1/2 \\ \ell_2 \geq m_1/2 + m_{12}^+ - m_{12} \\ \ell_2 \geq m_2/2 + (m_{12}^+ - m_{12}) \\ \qquad g_1 = m_{12} + m_{12}^+ + m_1 \\ \qquad g_2 = m_{12}^+ + m_1/2 + m_2/2 \end{array} \right. \qquad \begin{array}{l} n_1 \geq m_{12}^+ \\ n_1 \geq m_{12}^+ + m_1 - m_2 \\ \underline{n_1 \geq m_{12} + m_1 - m_2} \\ n_2 \geq m_2/2 \end{array}$$

f) $D_2, g = o_4 :$

$$(3.6) \left\{ \begin{array}{l} \underline{\ell_1 \geq m_{12}} \\ \ell_2 \geq m_{12}^+ \\ \qquad g_1 = m_{12} \\ \qquad g_2 = m_{12}^+ \end{array} \right. \qquad \begin{array}{l} n_1 \geq m_{12} \\ n_2 \geq m_{12}^+ \end{array}$$

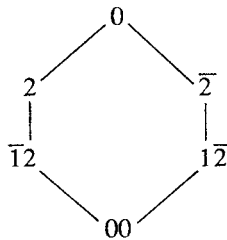
g) $D_3, g = o_6 :$

$$(3.7) \left\{ \begin{array}{l} \underline{\ell_1 \geq m_{12}} \\ \ell_2 \geq m_{13} \\ \ell_2 \geq m_{13} + m_{12}^+ - m_{12} \\ \ell_2 \geq m_{13} + m_{12}^+ - m_{12} + m_{23} - m_{13}^+ \\ \underline{\ell_3 \geq m_{13}^+} \\ \ell_3 \geq m_{13}^+ + m_{12}^+ - m_{12} \\ \ell_3 \geq m_{13}^+ + m_{12}^+ - m_{12} + m_{23}^+ - m_{13} \\ \qquad g_1 = m_{12} + m_{12}^+ + m_{13} + m_{13}^+ \\ \qquad g_2 = m_{12}^+ + m_{13} + m_{23} \\ \qquad g_3 = m_{12}^+ + m_{13}^+ + m_{23}^+ \end{array} \right. \qquad \begin{array}{l} n_1 \geq m_{12}^+ \\ n_1 \geq m_{12}^+ + m_{13}^+ - m_{23} \\ n_1 \geq m_{12}^+ + m_{13} - m_{23}^+ \\ n_1 \geq m_{12}^+ + m_{13} - m_{23}^+ + m_{13}^+ - m_{23} \\ \underline{n_1 \geq m_{12} + m_{13} - m_{23}^+ + m_{13}^+ - m_{23}} \\ \underline{n_2 \geq m_{23}} \\ n_3 \geq m_{23}^+ \end{array}$$

Proof. Parts a), b), c) are special cases of Theorem 2.1. To prove e) we use the result of [13]:

PROPOSITION 3.2 ([13], THEOREM 3). $c_{\ell_1, \ell_2; n_1, n_2}^{(sp_4)}(g_1, g_2)$ is equal to the number of 8-tuples $(p_2, p_{\bar{2}}, p_{\bar{1}}, p_{1\bar{2}}, p_{00}, p_{\bar{1}\bar{2}}, p_{\bar{1}\bar{2}}, p_0)$ of nonnegative integers satisfying the conditions:

(1) If s and t are two nonadjacent vertices of the graph



then $\min(p_s, p_t) = 0$.

$$(2) \begin{cases} \ell_1 \geq p_2 + p_{\bar{2}} + p_{\bar{1}} + 2p_0 & n_1 \geq p_2 + p_{\bar{1}} + 2p_{\bar{1}\bar{2}} + p_{00} \\ \ell_2 \geq p_{1\bar{2}} + p_{00} + p_{\bar{1}\bar{2}} + p_{\bar{1}\bar{2}} & n_2 \geq p_{\bar{2}} + p_0 + p_{1\bar{2}} + p_{\bar{1}\bar{2}} \\ g_1 = p_2 + p_{\bar{2}} + 2p_0 + 2p_{\bar{1}} + 2p_{\bar{1}\bar{2}} + p_{00} + 2p_{\bar{1}\bar{2}} \\ g_2 = p_{\bar{2}} + p_0 + p_{\bar{1}} + p_{1\bar{2}} + p_{\bar{1}\bar{2}} + p_{00} + 2p_{\bar{1}\bar{2}} \end{cases}$$

By this proposition to prove e) it suffices to construct a bijection between 8-tuples (p_s) satisfying (1) and (2), and sp_4 -partitions m satisfying (3.5). It is straightforward to verify that the desired bijection can be defined by formulas

$$\begin{cases} m_{12} = p_2 + p_{\bar{2}} + p_{\bar{1}} + 2p_0 \\ m_{1\bar{2}}^+ = p_{\bar{1}} + p_{00} \\ m_1 = 2(p_{\bar{1}\bar{2}} + p_{\bar{1}\bar{2}}) \\ m_2 = 2(p_{\bar{2}} + p_0 + p_{1\bar{2}} + p_{\bar{1}\bar{2}}) \\ p_{\bar{1}} = \min(m_{12}, m_{1\bar{2}}^+) \\ p_{\bar{1}\bar{2}} = \min(m_1/2, m_2/2) \\ p_{00} = [-\Delta_{12}]_+ \\ p_{\bar{1}\bar{2}} = [\Delta_{1\bar{2}}/2]_+ \\ p_2 = [\min(\Delta_{12}, \Delta_{12} + \Delta_{1\bar{2}})]_+ \\ p_{1\bar{2}} = [\min(-\Delta_{1\bar{2}}/2, -\Delta_{1\bar{2}}/2 - \Delta_{12})]_+ \\ p_0 = [\min(-\Delta_{1\bar{2}}/2, \Delta_{1\bar{2}}/2 + \Delta_{12})]_+ \\ p_{\bar{2}} = [\min(\Delta_{12}, -\Delta_{12} - \Delta_{1\bar{2}})]_+, \end{cases}$$

where $\Delta_{12} = m_{12} - m_{1\bar{2}}^+$, $\Delta_{1\bar{2}} = m_1 - m_2$ (see (2.3)), and $[x]_+ = \max(0, x)$.

Proof of d). Since the root systems B_2 and C_2 are isomorphic we have a (so called exceptional) isomorphism between o_5 and sp_4 . This isomorphism and the property $c_{\lambda\nu}(\gamma) = c_{\nu\lambda}(\gamma)$ (see §1) imply

$$c_{\ell_1\ell_2; n_1 n_2}^{(o_5)}(g_1, g_2) = c_{n_2 n_1; \ell_2 \ell_1}^{(sp_4)}(g_2, g_1).$$

Taken into account the already proven part e) it remains to construct a bijection between o_5 -partitions $(m_{12}, m_{1\bar{2}}^+, m_1, m_2)$ satisfying (3.4) and sp_4 -partitions $(\bar{m}_{12}, \bar{m}_{1\bar{2}}^+, \bar{m}_1, \bar{m}_2)$ satisfying

$$\begin{cases} \ell_1 \geq \bar{m}_2/2 & n_1 \geq \bar{m}_1/2 \\ \ell_2 \geq \bar{m}_{1\bar{2}}^+ & n_1 \geq \bar{m}_1/2 + \bar{m}_{1\bar{2}}^+ - \bar{m}_{12} \\ \ell_2 \geq \bar{m}_{1\bar{2}}^+ + \bar{m}_1 - \bar{m}_2 & n_1 \geq \bar{m}_2/2 + \bar{m}_{1\bar{2}}^+ - \bar{m}_{12} \\ \ell_2 \geq \bar{m}_{12} + \bar{m}_1 - \bar{m}_2 & n_2 \geq \bar{m}_{12} \\ g_1 = \bar{m}_{12} + \bar{m}_1/2 + \bar{m}_2/2 \\ g_2 = \bar{m}_{12} + \bar{m}_{1\bar{2}}^+ + \bar{m}_1 \end{cases}$$

(cf. (3.5)). Evidently, the desired bijection can be defined by $(m_{12}, m_{12}^+, m_1, m_2) = (\overline{m}_2/2, \overline{m}_1/2, \overline{m}_{12}^+, \overline{m}_{12})$.

Part f) is evident from the isomorphism $D_2 = A_1 \times A_1$. To prove g) we use the exceptional isomorphism $D_3 = A_3$ which implies

$$c_{\ell_1 \ell_2 \ell_3; n_1 n_2 n_3}^{(o_6)}(g_1, g_2, g_3) = c_{\ell_2 \ell_1 \ell_3; n_2 n_1 n_3}^{(sl_4)}(g_2, g_1, g_3)$$

Using c) it remains to construct a bijection between o_6 -partitions $(m_{12}, m_{23}, m_{13}, m_{12}^+, m_{23}^+, m_{13}^+)$ satisfying (3.7) and sl_4 -partitions $(\overline{m}_{12}, \overline{m}_{13}, \overline{m}_{14}, \overline{m}_{23}, \overline{m}_{24}, \overline{m}_{34})$ satisfying

$$\left\{ \begin{array}{ll} \ell_1 \geq \overline{m}_{13} & n_1 \geq \overline{m}_{24} \\ \frac{\ell_1 \geq \overline{m}_{13} + \overline{m}_{23} - \overline{m}_{12}}{\ell_2 \geq \overline{m}_{12}} & \frac{n_1 \geq \overline{m}_{24} + \overline{m}_{23} - \overline{m}_{34}}{n_2 \geq \overline{m}_{14}} \\ \frac{\ell_2 \geq \overline{m}_{12}}{\ell_3 \geq \overline{m}_{14}} & n_2 \geq \overline{m}_{14} + \overline{m}_{13} - \overline{m}_{24} \\ \ell_3 \geq \overline{m}_{14} + \overline{m}_{24} - \overline{m}_{13} & \frac{n_2 \geq \overline{m}_{14} + \overline{m}_{13} - \overline{m}_{24} + \overline{m}_{12} - \overline{m}_{23}}{n_3 \geq \overline{m}_{34}} \\ \ell_3 \geq \overline{m}_{14} + \overline{m}_{24} - \overline{m}_{13} + \overline{m}_{34} - \overline{m}_{23} & \\ & g_1 = \overline{m}_{13} + \overline{m}_{14} + \overline{m}_{23} + \overline{m}_{24} \\ & g_2 = \overline{m}_{12} + \overline{m}_{13} + \overline{m}_{14} \\ & g_3 = \overline{m}_{14} + \overline{m}_{24} + \overline{m}_{34} \end{array} \right.$$

(cf. (3.3)). One can verify that the following formulas define the desired bijection:

$$(3.9) \left\{ \begin{array}{ll} m_{12} = \overline{m}_{13} + [-\overline{\Delta}_{12}]_+ & \overline{m}_{12} = m_{13} + [-\Delta_{12} + [-\Delta_{1\overline{2}}]_+]_+ \\ m_{12}^+ = \min(\overline{m}_{24}, \overline{m}_{13} + [\overline{\Delta}_{12}]_+) & \overline{m}_{23} = m_{13} + [\Delta_{12} - [-\Delta_{1\overline{2}}]_+]_+ \\ m_{23} = \overline{m}_{14} + [\overline{\Delta}_{13} + [\overline{\Delta}_{12}]_+]_+ & \overline{m}_{34} = m_{23}^+ \\ m_{23}^+ = \overline{m}_{34} & \overline{m}_{13} = \min(m_{12}, m_{12}^+ + [-\Delta_{1\overline{2}}]_+) \\ m_{13} = \min(\overline{m}_{12}, \overline{m}_{23}) & \overline{m}_{24} = m_{12}^+ + [\Delta_{1\overline{2}}]_+ \\ m_{13}^+ = \overline{m}_{14} + [-\overline{\Delta}_{13} - [\overline{\Delta}_{12}]_+]_+ & \overline{m}_{14} = \min(m_{13}^+, m_{23}), \end{array} \right.$$

where $\Delta_{12} = m_{12} - m_{12}^+$, $\Delta_{1\overline{2}} = m_{13}^+ - m_{23}$ (cf. (2.3)), $\overline{\Delta}_{12} = \overline{m}_{12} - \overline{m}_{23}$, $\overline{\Delta}_{13} = \overline{m}_{13} - \overline{m}_{24}$ (cf. (2.1)). ■

4. REDUCTION MULTIPLICITIES AND GENERALIZED GELFAND-TSETLIN PATTERNS

THEOREM 4.1. *Let g be a simple classical Lie algebra of rank r , and $\lambda = \sum_{1 \leq i \leq r} \ell_i \omega_i$ a highest weight for g . Then the weight multiplicity $K_{\lambda\beta}$ is equal to the number of g -partitions m of weight $\lambda - \beta$ satisfying $\max_s \ell_j^{(s)}(m) \leq \ell_j$ for $j = 1, \dots, r$, where the $\ell_j^{(s)}$ were defined in (2.2), (2.4), (2.6).*

Finally, an o_{2r} -pattern is an array $(\lambda_{ij} (1 \leq i \leq j \leq r), \eta_{ij} (1 \leq i \leq j \leq r-1))$ satisfying.

- (1) Either all λ_{ij} and η_{ij} are integers or all lie in $\frac{1}{2} + \mathbb{Z}$.
- (2) $\min(\lambda_{i,j-1}, \lambda_{i+1,j-1}) \geq \eta_{i,j-1} \geq \max(\lambda_{ij}, \lambda_{i+1,j})$ for $1 \leq i < j \leq r$, and $\lambda_{i,r} + \lambda_{i+1,r} + \min(\lambda_{i,r-1}, \lambda_{i+1,r-1}) \geq \eta_{i,r-1}$ for $1 \leq i \leq r-1$ (with the convention $\lambda_{i+1,i} = +\infty$).

The *highest weight* of each of these patterns is the weight $\lambda_{11}\epsilon_1 + \lambda_{12}\epsilon_2 + \dots + \lambda_{1r}\epsilon_r$, and the weight $\beta = \beta_1\epsilon_1 + \dots + \beta_r\epsilon_r$ of a pattern is defined by $\beta_i = |\lambda_i| - 2|\eta_i| + |\lambda_{i+1}|$. It is easy to see that the highest weight of a g -pattern Λ is a highest g -weight, and the weight of Λ is an integral g -weight.

It is also convenient to identify a gl_r -pattern $\Lambda = (\lambda_{ij})$ with an o_{2r+1} , sp_{2r} , or o_{2r} -pattern $(\lambda_{ij}, \eta_{ij})$, where $\eta_{ij} = \lambda_{i+j,j+1}$ ($1 \leq i \leq j \leq r-1$) and $\eta_{i,r} = 0$. Clearly, this identification preserves highest weight and weight.

We shall show that Theorem 4.1 is equivalent to

THEOREM 4.2. *For any simple classical Lie algebra g the weight multiplicity $K_{\lambda\beta}$ is equal to the number of all g -patterns Λ of highest weight λ and weight β .*

To show that Theorems 4.1 and 4.2 are equivalent it suffices to construct a bijection between g -partitions m from Theorem 4.1 and g -patterns Λ from Theorem 4.2. It is easy to verify that such bijection in each case can be defined by formulas

$$(4.1) \quad m_{ij} = \eta_{i,j-1} - \lambda_{ij}, \quad m_{ij}^+ = \eta_{i,j-1} - \lambda_{i+1,j}, \quad m_i = 2\eta_{i,r}.$$

For $g = sl_r$ the statement of Theorem 4.2 is well-known (the simple proof using only (0.1) is given in [6]). Before considering other algebras we shall outline the proof of Theorem 2.1.

Proof of Theorem 2.1. For $g = sl_r$ the correspondence (4.1) takes form $m_{ij} = \lambda_{i+1,j} - \lambda_{ij}$. It is easy to see that it transforms the $n_i^{(t)}(m)$ from (2.2) into

$$(4.2) \quad n_i^{(t)}(\Lambda) = \lambda_{i+1,t} - \lambda_{it} + \sum_{t+1 \leq p \leq r} (2\lambda_{i+1,p} - \lambda_{ip} - \lambda_{i+2,p}).$$

Therefore, Theorem 2.1 is equivalent to

THEOREM 4.3. *Let λ, μ and $\nu = \sum_{1 \leq i \leq r-1} n_i \omega_i$ be highest sl_r -weights. Then $c_{\lambda\nu}^\mu$ is equal to the number of sl_r -patterns Λ of highest weight λ , weight $(\mu - \nu)$ and such that $n_i^{(t)}(\Lambda) \leq n_i$ for $1 \leq i \leq r-1, i < t \leq r$.*

But Theorem 4.3 is essentially proven in [2], [6] (note that definition of the weight of a GT-pattern used there differs from the present one by the transformation $(\beta, \dots, \beta_r) \rightarrow (\beta_r, \dots, \beta_1)$).

REMARK. Correspondence (4.1) enables one to translate Conjecture 2.2 to the language of \mathfrak{g} -patterns. We leave this reformulation to the reader.

The proof of Theorem 4.2 becomes easier if we generalize it to some reduction multiplicities. Fix one of the series B, C, D and denote the corresponding Lie algebra of rank r by \mathfrak{g}_r . For $k = 0, \dots, r$ consider the subset of simple roots $\Pi' = \{\alpha_{k+1}, \dots, \alpha_r\}$. Clearly, the corresponding semisimple Lie subalgebra $\mathfrak{g}_0(\Pi')$ is isomorphic to \mathfrak{g}_{r-k} , and the Levi subalgebra $\mathfrak{g}(\Pi')$ naturally decomposes as $\mathbb{C}^k \oplus \mathfrak{g}_{r-k}$ (see §1). A $\mathfrak{g}(\Pi')$ -weight will be thought of as a pair (β', λ') , where $\beta' = (\beta'_1, \dots, \beta'_k)$, and $\lambda' = (\lambda'_{k+1}, \dots, \lambda'_r)$ is a \mathfrak{g}_{r-k} -weight. Denote by $K_{\lambda; \beta', \lambda'}$ the multiplicity of an irreducible $\mathfrak{g}(\Pi')$ -module $V'_{(\beta', \lambda')}$ in the reduction $V_\lambda|_{\mathfrak{g}(\Pi')}$ (see §1); for $k = r$ this is just the weight multiplicity $K_{\lambda \beta}$.

For any $k = 0, 1, \dots, r$ by a truncated \mathfrak{g}_r -pattern Λ we mean an array $(\lambda_{ij} \ (1 \leq i \leq k+1, i \leq j \leq r), \eta_{ij} \ (1 \leq i \leq k, i \leq j \leq r))$ which can be extended to a \mathfrak{g}_r -pattern. For example, for $k = 1$ a truncated \mathfrak{g}_r -pattern looks as

$$\begin{matrix} \lambda_{11} & \lambda_{12} & \dots & \lambda_{1r} \\ & \eta_{11} & \eta_{12} & \dots & \eta_{1r} \\ & & \lambda_{22} & \dots & \lambda_{2r} \end{matrix},$$

and for $k = r$ it is a usual \mathfrak{g}_r -pattern. The highest weight of a truncated \mathfrak{g}_r -pattern Λ is $\lambda = (\lambda_{11}, \lambda_{12}, \dots, \lambda_{1r})$, and the weight of Λ is the pair (β', λ') , where $\beta'_i = |\lambda_i| - 2|\eta_i| + |\lambda_{i+1}| \ (1 \leq i \leq k)$ and $\lambda' = (\lambda_{k+1, k+1}, \lambda_{k+1, k+2}, \dots, \lambda_{k+1, r})$.

THEOREM 4.4. *The reduction multiplicity $K_{\lambda; \beta', \lambda'}$ is equal to the number of truncated \mathfrak{g}_r -patterns of highest weight λ and weight (β', λ') .*

An obvious induction shows that it is sufficient to prove Theorem 4.4 for $k = 1$, i.e. to compute reduction of irreducible modules from \mathfrak{g}_r to $\mathbb{C} \oplus \mathfrak{g}_{r-1}$. The reduction from sp_{2r} to $(\mathbb{C} \oplus sp_{2r-2})$ was computed by D.P. Zhelobenko [9], and the case of orthogonal Lie algebras can be deduced from the results of [17]. Another way to prove Theorem 4.4 for $k = 1$ is to use the involution method of [6]. Details will be given in another publication.

REMARKS. 1. The similar result holds for $\mathfrak{g} = s\ell_r$. In this case it is well-known (see e.g. [6]).

2. It is easy to verify that Theorem 4.4 is compatible with Conjecture 2.2.

REFERENCES

- [1] I.M. GELFAND, A.V. ZELEVINSKY. *Polytopes in the pattern space and canonical basis in irreducible representations of gl_3* . *Funct. Anal and Appl.* (in Russian), 1985, **19**, No. 2, 72-75.
- [2] I.M. GELFAND, A.V. ZELEVINSKY. *Multiplicities and regular bases for gl_n* . in «Group theoretical methods in physics», Proc. of the third seminar, Yurmala, May 22-24, 1985, v.2. 22-31. Moscow, Nauka 1986.
- [3] I.M. GELFAND, A.V. ZELEVINSKY. Canonical basis in irreducible representations of gl_3 and its applications, *ibid*, 31-45.
- [4] F.A. BEREZIN, I.M. GELFAND. *Some remarks on the theory of spherical functions on Riemannian symmetric manifolds*. Proc. Moscow Math. Soc. (in Russian), 1956, **5**, 311-352.
- [5] B. KOSTANT. *A formula for the multiplicity of a weight*. *Trans. Amer. Math. Soc.*, 1959, **93**, 53-73.
- [6] A.D. BERENSTEIN, A.V. ZELEVINSKY. *Involutions on Gelfand-Tsetlin patterns and multiplicities in skew GL_n -modules*. *Doklady Akad. Nauk USSR*, 1988, **300**, No. 6, 1291-1294.
- [7] D.E. LITTLEWOOD. *The theory of group characters*. 2nd edn. Oxford Univ. Press., 1950.
- [8] I.G. MACDONALD. *Symmetric functions and Hall polynomials*. Clarendon Press, Oxford, 1979.
- [9] D.P. ZHELOBENKO. *Compact Lie groups and their representations*. (in Russian). Moscow, Nauka, 1970.
- [10] A.V. ZELEVINSKY. *Multiplicities in finite dimensional representations of semisimple Lie algebras*. *Uspekhi Math. Nauk*, 1989.
- [11] K. BACLAWSKI. *A new rule for computing Clebsch-Gordan series*. *Adv. in Appl. Math.*, 1984, **5**, 416-432.
- [12] C. DE CONCINI, D. KAZHDAN. *Special bases for S_n and GL_n* . *Israel.J. Math.*, 1981, **40**, 275-290.
- [13] V.S. RETAKH, A.V. ZELEVINSKY. *Base affine space and canonical basis in irreducible representations of Sp_4* . *Doklady Akad. Nauk USSR*, 1988, **300**, No. 1, 31-35.
- [14] J.D. LOUCK, L.C. BIEDENHARN. *Some properties of the intertwining number of the general linear group*. in «Science and computers», *Adv. in Math. Suppl. Studies*, 1986, **10**, 265-311.
- [15] R.C. KING, N.G.I. EL - SHARKAWAY. *Standard Young tableaux and weight multiplicities of the classical Lie groups*. *J. Phys. A.*, 1983, **16**, 3153-3178.
- [16] G.R. BLACK, R.C. KING, B.G. WYBOURNE. *Kronecker product for compact semisimple Lie groups*. *J. Phys. A.*, 1983, **16**, 1555-1589.
- [17] R.C. KING. *Branching rules for classical Lie groups using tensor and spinor methods*. *J. Phys. A.*, 1975, **8**, 429-449.
- [18] K. KOIKE, I. TERADA. *Young-diagrammatic methods for the representation theory of the classical groups of type B_n, C_n, D_n* . *J. Algebra*, 1987, **107**, No. 2, 466-511.
- [19] V. LAKSHMIBAI, C.S. SESHADRI. *Geometry of $G/P, V$* . *J. Algebra*, 1986, **100**, 462-557.
- [20] P. LITTELMANN. *A generalization of the Littlewood-Richardson rule*, Dec. 1987, Preprint.
- [21] N. BOURBAKI. *Groupes et Algèbres de Lie*. Ch. IV-VI. Herman, Paris, 1968.

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